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The Fractional Fokker-Planck Equation Analysis with the Caputo-Fabrizio Operator

Mohammed A. Hussein[1], Hossein Ali Eaued[2], Ahmed Baqer Jaafer[3]

[1,2]Scientific Research Center, Thi-Qar University, Thi-Qar, Iraq.

[3]Faculty of Education for Pure Sciences, University of Thi-Qar, Nasiriyah, Iraq.

ABSTRACT: This study used the Daftardar-Jafari approach to find the approximate and analytical solution for the Fokker-Planck (F-P) equation with the operator Caputo-Fabrizio (DJM). The accuracy, efficiency, and simplicity of the current technique are excellent.

KEYWORDS: Daftardar-Jafari method; Fokker-Planck equation; fractional differential equation, Caputo -Fabrizio operator.

1. INTRODUCTION

Numerous theoretical and practical sciences, such as theoretical biology and ecology, solid-state mechanics, viscosity, optical fibers, data analysis, electrical control theory, stochastic economics, hydrodynamics, and dynamics, among others, significantly rely on the computation and analysis of nonlinear partial differential equation solutions. This process has been going on for at least half a century [1–3]. Numerous analytical and numerical techniques have been used recently to try and solve fractional differential equations (FDEs). Since most fractional differential equations lack exact solutions, approximating and numerical techniques are utilized to solve the FDEs. [4,5]. In this work, we use DJM to solve Fokker-Planck equation that include the fractional operator of the Caputo-Fabrizio type. One of the fundamental equations in the study of stochastic processes, like the Markov process, has been the Fokker-Planck (F-P) equation [6]. The transition probability density function $\psi(\mu,\mathfrak{t})$ is described by and is represented by this parabolic differential equation.

$$\frac{\partial\psi}{\partial\mathfrak{t}} = -\frac{\partial}{\partial\mu}\{A(\mu)\psi\} + \frac{1}{2}\frac{\partial^2}{\partial\mu^2}\{B(\mu)\psi\}, \qquad (1)$$

Where coefficient A is known as the drifting term and coefficient $B \geq 0$ is known as the fluctuation factor.
In this study, the DJ method was applied to the Fractional-order Fokker-Planck equation, where the general form of this equation is [7],

$$\frac{\partial^\lambda\psi}{\partial\mathfrak{t}^\lambda} = \frac{1}{2}\frac{\partial^2}{\partial\mu^2}\{B(\mu)\psi\} - \frac{\partial}{\partial\mu}\{A(\mu)\psi\}, 0<\lambda<1, \mu\in R, \mathfrak{t}>0. \qquad (2)$$

2. PRELIMINARIES OF FRACTIONAL CALCULUS

DEFINITION 1 [8–11]. let $\psi \in H^1(\varrho,\sigma), \varrho > \sigma, \varrho \in (-\infty,\mathfrak{t}), 0<\lambda<1$, then, the definition of the Caputo-Fabrizio fractional derivative is

$${}^{\mathrm{CF}}_{\varrho}D^{\lambda}_{\mathfrak{t}}\psi(\mathfrak{t}) = \frac{\beta(\lambda)}{(1-\lambda)}\int_{\varrho}^{\mathfrak{t}}\psi'(s)\exp\left(-\frac{\lambda}{1-\lambda}(\mathfrak{t}-s)\right)ds \qquad (3)$$

Where $\beta(\lambda)$ is a normalizing function that satisfies $\beta(0)=\beta(1)=1$.
The operator's fundamental characteristics are as follows:

1 ${}^{\mathrm{CF}}D^{\lambda}_{\mathfrak{t}}\psi(\mathfrak{t}) = \psi(\mathfrak{t})$, where $\lambda = 0$.
2 ${}^{\mathrm{CF}}D^{\lambda}_{\mathfrak{t}}[\psi(\mathfrak{t})+\phi(\mathfrak{t})] = {}^{\mathrm{CF}}D^{\lambda}_{\mathfrak{t}}\psi(\mathfrak{t}) + {}^{\mathrm{CF}}D^{\lambda}_{\mathfrak{t}}\phi(\mathfrak{t})$.
3 ${}^{\mathrm{CF}}D^{\lambda}_{\mathfrak{t}}(c) = 0$, where c is constant.

DEFINITION 2 [5, 12, 13]. let $\psi \in H^1(\varrho,\sigma), \varrho > \sigma, \varrho \in (-\infty, \mathfrak{r}), 0 < \lambda < 1$, then, the fractional integral of order α of a function u is defined by

$$ {}^{\mathrm{CF}}_{\varrho}I^{\lambda}_{\mathfrak{r}}\psi(\mathfrak{r}) = \frac{1-\lambda}{\beta(\lambda)}\psi(\mathfrak{r}) + \frac{\lambda}{\beta(\lambda)}\int_{\varrho}^{\mathfrak{r}}\psi(s)ds, \tag{4}$$

Where $\beta(\lambda)$ is a normalizing function that satisfies $\beta(0) = \beta(1) = 1$.

1. The operator's fundamental characteristics are as follows:
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3. ${}^{\mathrm{CF}}I^{\lambda}_{\mathfrak{r}}[\psi(\mathfrak{r}) + \phi(\mathfrak{r})] = {}^{\mathrm{CF}}I^{\lambda}_{\mathfrak{r}}\psi(\mathfrak{r}) + {}^{\mathrm{CF}}I^{\lambda}_{\mathfrak{r}}\phi(\mathfrak{r})$
4. ${}^{\mathrm{CF}}I^{\lambda}_{\mathfrak{r}}[{}^{\mathrm{CF}}D^{\lambda}_{\mathfrak{r}}\psi(\mathfrak{r})] = \psi(\mathfrak{r}) - \psi(0)$

3. THE TECHNIQUE ANALYSIS

Let's suppose that Eq.(2) with $\psi(\mu,0) = \psi_0(\mu)$. The typical format of Eq.(2) in Caputo-Fabrizio sense is

$$ {}^{\mathrm{CF}}D^{\lambda}_{\mathfrak{r}}\psi = \frac{1}{2}\frac{\partial^2}{\partial\mu^2}\{B(\mu)\psi\} - \frac{\partial}{\partial\mu}\{A(\mu)\psi\} \tag{5}$$

with initial conditions

$$\psi(\mu,0) = \psi_0(\mu)$$

where ${}^{\mathrm{CF}}D^{\lambda}_{\mathfrak{r}}\psi(\mu,\mathfrak{r})$ is Caputo-Fabrizio operator of $\psi(\mu,\mathfrak{r}), 0 < \lambda \leq 1$.

The result below is obtained by applying the Caput-Fabrizio integral to both sides of Eq.(5).

$$ {}^{\mathrm{CF}}I^{\lambda}_{\mathfrak{r}}[{}^{\mathrm{CF}}D^{\lambda}_{\mathfrak{r}}\psi(\mu,\mathfrak{r})] = {}^{\mathrm{CF}}I^{\lambda}_{\mathfrak{r}}\left[\frac{1}{2}\frac{\partial^2}{\partial\mu^2}\{B(\mu)\psi\}\right] - {}^{\mathrm{CF}}I^{\lambda}_{\mathfrak{r}}\left[\frac{\partial}{\partial\mu}\{A(\mu)\psi\}\right], \tag{6}$$

then, we obtain

$$\psi(\mu,\mathfrak{r}) = \psi(\mu,0) + {}^{\mathrm{CF}}I^{\lambda}_{\mathfrak{r}}\left[\frac{1}{2}\frac{\partial^2}{\partial\mu^2}\{B(\mu)\psi\}\right] - {}^{\mathrm{CF}}I^{\lambda}_{\mathfrak{r}}\left[\frac{\partial}{\partial\mu}\{A(\mu)\psi\}\right]. \tag{7}$$

We are trying to find a series-formable solution to Eq.(7),

$$\psi(\mu,\mathfrak{r}) = \sum_{n=0}^{\infty} \psi_n(\mu,\mathfrak{r}), \tag{8}$$

When Eq.(7) is changed to reflect the decomposition series Eq.(8), it produces

$$\sum_{n=0}^{\infty} \psi_n(\mu,\mathfrak{r}) = \psi_0(\mu) + {}^{\mathrm{CF}}I^{\lambda}_{\mathfrak{r}}\left[\frac{1}{2}\frac{\partial^2}{\partial\mu^2}\{B(\mu)(\textstyle\sum_{n=0}^{\infty}\psi_n)\}\right] - {}^{\mathrm{CF}}I^{\lambda}_{\mathfrak{r}}\left[\frac{\partial}{\partial\mu}\{A(\mu)(\textstyle\sum_{n=0}^{\infty}\psi_n)\}\right]. \tag{9}$$

Additionally, recurrence is used to describe the relationship so that

$$\begin{aligned}
&\psi_0(\mu,\mathfrak{r}) = \psi_0(\mu),\\
&\psi_{n+1}(\mu,\mathfrak{r}) = {}^{\mathrm{CF}}I^{\lambda}_{\mathfrak{r}}\left[\frac{1}{2}\frac{\partial^2}{\partial\mu^2}\{B(\mu)(\psi_n)\}\right] - {}^{\mathrm{CF}}I^{\lambda}_{\mathfrak{r}}\left[\frac{\partial}{\partial\mu}\{A(\mu)(\psi_n)\}\right].
\end{aligned} \tag{10}$$

The approximate k-term solution of Eq. (5) is thus provided by:

$$\psi(\mu,\mathfrak{r}) = \psi_0(\mu,\mathfrak{r}) + \psi_1(\mu,\mathfrak{r}) + \psi_2(\mu,\mathfrak{r}) + \psi_3(\mu,\mathfrak{r}) + \cdots. \tag{11}$$

4. APPLICATION

Take into account the time fractional F-P equation of order with $\psi(\mu,0) = \mu$. We choose the constant fluctuation term $B(\mu) = 2k$ and the drift term $A(\mu) = \mu$ for simplicity. Using the relationship Eq.(10), we immediately

$$\begin{aligned}
\psi_0 &= \mu,\\
\psi_1 &= {}^{\mathrm{CF}}I^{\lambda}_{\mathfrak{r}}\left[\frac{1}{2}\frac{\partial^2}{\partial\mu^2}\{2k\psi_0\}\right] - {}^{\mathrm{CF}}I^{\lambda}_{\mathfrak{r}}\left[\frac{\partial}{\partial\mu}\{-\mu\psi_0\}\right]\\
&= 2\mu(1-\lambda+\lambda\mathfrak{r}),\\
\psi_2 &= {}^{\mathrm{CF}}I^{\lambda}_{\mathfrak{r}}\left[\frac{1}{2}\frac{\partial^2}{\partial\mu^2}\{2k\psi_1\}\right] - {}^{\mathrm{CF}}I^{\lambda}_{\mathfrak{r}}\left[\frac{\partial}{\partial\mu}\{-\mu\psi_1\}\right]\\
&= 4\mu\left[(1-2\lambda+\lambda^2) + (2\lambda-2\lambda^2)\mathfrak{r} + \frac{1}{2}\lambda^2\mathfrak{r}^2\right],\\
\psi_3 &= {}^{\mathrm{CF}}I^{\lambda}_{\mathfrak{r}}\left[\frac{1}{2}\frac{\partial^2}{\partial\mu^2}\{2k\psi_2\}\right] - {}^{\mathrm{CF}}I^{\lambda}_{\mathfrak{r}}\left[\frac{\partial}{\partial\mu}\{-\mu\psi_2\}\right]\\
&= 8\mu\left[(1-3\lambda+3\lambda^2-\lambda^3) + (1-3\lambda^2+2\lambda^3)\mathfrak{r} + \left(\lambda-\frac{1}{2}\lambda^2-\frac{1}{2}\lambda^3\right)\mathfrak{r}^3 + \frac{1}{6}\lambda^2\mathfrak{r}^3\right].
\end{aligned} \tag{12}$$

Now, the approximate solution of Eq.(5) is

$$\begin{aligned}\psi(\mu,ꞇ) = \quad & \mu + 2\mu(1-\lambda+\lambdaꞇ) + 4\mu\begin{bmatrix}(1-2\lambda+\lambda^2)\\ +(2\lambda-2\lambda^2)ꞇ + \frac{1}{2}\lambda^2ꞇ^2\end{bmatrix} \\ & +8\mu\begin{bmatrix}(1-3\lambda+3\lambda^2-\lambda^3)+(1-3\lambda^2+2\lambda^3)ꞇ \\ +\left(\lambda-\frac{1}{2}\lambda^2-\frac{1}{2}\lambda^3\right)ꞇ^3+\frac{1}{6}\lambda^2ꞇ^3\end{bmatrix}+\cdots. \end{aligned} \tag{13}$$

The Eq.(13) provides a rough solution to the form.,

$$\psi(\mu,ꞇ) = \mu e^{2ꞇ},$$

for $\lambda = 1$, which is the exact solution of Eq. (5) at $\lambda = 1$.

5. CONCLUSIONS

This method, which has demonstrated its effectiveness in solving these kinds of equations, is one of the most significant and current strategies for solving linear and nonlinear differential equations. We made a significant discovery in this study, the idea is those differential equations utilizing the Caputo-Fabrizio fractional operator may be effectively solved using this method. We solved the Fokker-Planck equation, one of the most important physics equations..

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