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Coefficient Estimate for a Subclass of Close-to-Convex Functions with Respect To Symmetric and Conjugate Points Connected With the Q–Borel Distribution

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ABSTRACT: The main purpose of this paper is to introduce a new subclass of close-to-convex functions in the open unit disc $\mathcal{U}$, denoted by $\kappa^{\alpha,\tau,q}(A,B)$ with respect to symmetric and conjugate points by applying a q-analogue of the familiar Borel distribution (BD), which is a subclass of all functions that are analytic, univalent and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. We find estimates $|a_2|$, $|a_3|$, $|a_4|$ and $|a_5|$ for Taylor-Maclaurin coefficients of the functions in the subclass introduced, and a brief discussion is also given to the pertinent relationship between these classes and the famous Fekete- Szegö theorem for $|a_3 - \mu a_2^2|$ and $|a_2 a_4 - a_3^2|$. Also, we deduce various corollaries and consequences of the main results when $q \to 1^-$. We find the sufficient condition for a function $f(z)$ to be in the class $\kappa_s^{\alpha,\tau,q}(A,B)$ and $\kappa_c^{\alpha,\tau,q}(A,B)$.

KEYWORDS-Analytic functions; Univalent functions; Coefficient estimates; Symmetric points; Conjugate points.

1. INTRODUCTION

Let $\mathcal{A}$ be the class of functions f which are analytic, univalent and normalized with $f(0) = 0$ and $f'(0) = 1$, the function f is given by

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j, \quad z \in \mathcal{U}, \tag{1.1}$$

where

$\mathcal{U} := \{z \in \mathbb{C} : |z| < 1\}$,

is the unit disc. Also, the function $g \in \mathcal{A}$ is given by

$$g(z) = z + \sum_{j=2}^{\infty} b_j z^j, \quad z \in \mathcal{U}. \tag{1.2}$$

Therefore, the functions $f(z)$ and $g(z)$ have a convolution (or Hadamard product), which is given by

$$f(z) * g(z) = (f * g)(z) = z + \sum_{j=2}^{\infty} a_j b_j z^j, z \in \mathcal{U}. \tag{1.3}$$

Consider Ω to be the family of functions $w(z)$ which are analytic in $\mathcal{U}$ and defined as follows

$$\Omega = \{w \in \mathcal{A} : w(0) = 0 \text{ and } |w(z)| < 1, z \in \mathcal{U}\}. \tag{1.4}$$

Assume that $f(z), g(z) \in \mathcal{A}$, we say that f is subordinate to g (written $f(z) \prec g(z)$) if there exists a Schwarz function $w(z)$, analytic such that

$f(z) = g(w(z)) \; (z \in \mathcal{U})$.

In particular, if the function $g(z)$ is univalent in $\mathcal{U}$, then the subordinate is equivalent to (see [2, 13])

$$f(0) = g(0) \text{ and } f(\mathfrak{U}) \subset g(\mathfrak{U}). \tag{1.5}$$

Definition 1. [10] For A and B are arbitrarily fixed numbers and that $(-1 \leq B < A \leq 1)$, denote the family by $P[A, B]$ containing functions of the form

$$P(z) = 1 + \sum_{j=1}^{\infty} p_j z^j. \tag{1.6}$$

is analytic in $\mathfrak{U}$ and then

$$P(z) \prec \frac{1 + Az}{1 + Bz} \text{ or } P(z) = \frac{1 + A\,w(z)}{1 + B\,w(z)}, (z \in \mathfrak{U}) \tag{1.7}$$

holds.

In the year 1973, Janowski [10] introduced the following subclass of starlike functions:

$$S^*(A, B) := \left\{ f : f \in \mathcal{A} \text{ and } \frac{zf'(z)}{f(z)} \prec \frac{1 + Az}{1 + Bz} \right\},$$

for $(-1 \leq B < A \leq 1;\ z \in \mathfrak{U})$.

On the other hand, in 1959, Sakaguchi [19] introduced the class of starlike functions with respect to symmetric points as follows:

$$S_s^* := \left\{ f : f \in \mathcal{A} \text{ and } \Re\left(\frac{zf'(z)}{f(z) - f(-z)} \right) > 0;\ z \in \mathfrak{U} \right\}.$$

Goel and Mehrok [8] introduced the subclass $S_s^*(A, B)$ of class S_s^* as follows:

$$S_s^*(A, B) := \left\{ f : f \in \mathcal{A} \text{ and } \frac{2zf'(z)}{f(z) - f(-z)} \prec \frac{1 + Az}{1 + Bz} \right\}$$

for $(-1 \leq B < A \leq 1;\ z \in \mathfrak{U})$.

El-Ashwah and Thomas [4] introduced the class of starlike functions with respect to conjugate points as follows:

$$S_c^* := \left\{ f : f \in \mathcal{A} \text{ and } \Re\left(\frac{zf'(z)}{f(z) + \overline{f(\bar{z})}} \right) > 0;\ z \in \mathfrak{U} \right\}.$$

Many authors introduced the analogue definitions by extension as follows (see [3]):

$$S_c^*(A, B) := \left\{ f : f \in \mathcal{A} \text{ and } \frac{2zf'(z)}{f(z) + \overline{f(\bar{z})}} \prec \frac{1 + Az}{1 + Bz};\ z \in \mathfrak{U} \right\}.$$

Since $S_c^*(A, B)$ is the subclass of starlike functions with respect to conjugate points. More results can be found in [21, 22, 23] on starlike functions with respect to symmetric and conjugate points. In 1952, Kaplan [12] introduced the class of close-to-convex functions as follows

$$\kappa := \left\{ f : f \in \mathcal{A} \text{ and } \Re\left(\frac{zf'(z)}{g(z)} \right) > 0;\ z \in \mathfrak{U} \right\}.$$

Let $\kappa_s(A, B)$ be the class of close-to-convex functions with respect to symmetric point (see [11])

$$\kappa_s(A, B) := \left\{ f : f \in \mathcal{A} \text{ and } \frac{2zf'(z)}{g(z) - g(-z)} \prec \frac{1 + Az}{1 + Bz};\ (-1 \leq B < A \leq 1;\ z \in \mathfrak{U}) \right\}, \tag{1.8}$$

where $g(z) = z + \sum_{j=2}^{\infty} b_j z^j \in S_s^*(A, B).$ (1.9)

Let $\kappa_c(A, B)$ be the class of close-to-convex functions with respect to conjugate point (see [27]) when $\alpha = 0$

$$\kappa_c(A, B) := \left\{ f : f \in \mathcal{A} \text{ and } \frac{2zf'(z)}{g(z) + \overline{g(\bar{z})}} \prec \frac{1 + Az}{1 + Bz};\ (-1 \leq B < A \leq 1;\ z \in \mathfrak{U}) \right\},$$

(1.10)

where $g(z) = z + \sum_{j=2}^{\infty} b_j z^j \in S_c^*(A, B).$ (1.11)

Coefficient Estimate for a Subclass of Close-to-Convex Functions with Respect To Symmetric and Conjugate Points Connected With the Q–Borel Distribution

Huo Tang and Guan-Tie Deng [27] introduced new subclasses of close-to-convex and quasiconvex functions with respect to symmetric and conjugate points. Such probability distributions as the Logarithmic, the Binomial, the Poisson, the Pascal, and other distributions have recently alleared in different context in the Geometric Function Theory of Complex Analysis mainly from a theoretical viewpoint (see [1, 5, 15, 17]). Recently, Wanas and Khuttar [28] have presented the Borel distribution (BD) whose probability mass function is given as follow

$$\text{Prob}\{x = \rho\} = \frac{(\rho\tau)^{\rho-1} e^{-\rho\tau}}{\rho!} \quad (\rho = 1,2,3,\dots).$$

Recall that if a discrete random variable x takes on the values $1, 2, 3, \dots$ with the following probabilities, it is said to have a Borel distribution:

$$\frac{e^{-\tau}}{1!}, \frac{2\tau e^{-2\tau}}{2!}, \frac{9\tau^2 e^{-3\tau}}{3!}, \dots, \qquad (1.12)$$

respectively, where τ is the parameter involved.

Wanas and Khuttar [28] also introduced the following series $\mu(\tau; z)$ whose coefficients are probabilities of the Borel distribution (BD):

$$\mu(\tau, z) := z + \sum_{j=2}^{\infty} \frac{[\tau(j-1)]^{j-2} e^{-\tau(j-1)}}{(j-1)!} z^j,$$

$$:= z + \sum_{j=2}^{\infty} \varphi_j(\tau) z^j \quad (0 < \tau \leq 1),$$

where, for convenience,

$$\varphi_j(\tau) := \frac{[\tau(j-1)]^{j-2} e^{-\tau(j-1)}}{(j-1)!}.$$

We now recall the following linear operator $\mathcal{Q}(\tau; z)$ for functions $f: \mathcal{A} \to \mathcal{A}$ (see [6, 14, 25]):as follows:

$$\mathcal{Q}(\tau; z) f(z) = \mu(\tau; z) * f(z)$$

$$= z + \sum_{j=2}^{\infty} \frac{[\tau(j-1)]^{j-2} e^{-\tau(j-1)}}{(j-1)!} a_j z^j \quad (0 < \tau \leq 1).$$

(1.13)

We now recall several concepts and notations of the classical q-calculus, which is primarily inspired by the work of Srivastava [24], who employed a variety of operators of q-calculus and fractional q-calculus. First, the q-Pochhammer symbol $(\tau; q)_n$ is defined, for $\tau, q \in \mathbb{C}$ and $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, by

$$(\tau; q)_n = \begin{cases} 1 & (n = 0) \\ (1-\tau)(1-\tau q)\dots(1-\tau q^{n-1}) & (n \in \mathbb{N}) \end{cases}$$

(1.14)

and

$$(\tau; q)_\infty = \prod_{j=0}^{\infty} (1 - \tau q^j) \qquad (|q| < 1).$$

According to the q-gamma function $\Gamma_q(z)$ defined by (see [7])

$$\Gamma_q(z) = (1-q)^{1-z} \frac{(q; q)_\infty}{(q^z; q)_\infty} \qquad (|q| < 1; z \in \mathbb{C}),$$

it is easily seen from (1.14) that

$$(q^\tau; q)_n = \frac{(1-q)^n \Gamma_q(\tau + n)}{\Gamma_q(\tau)} \quad (n \in \mathbb{N}_0)$$

The q –gamma function $\Gamma_q(z)$ is known to satisfy the following recurrence relation:

$$\Gamma_q(z+1) = [z]_q \Gamma_q(z),$$

where $[\tau]_q$ denotes the basic (or q-) number defined as follows:

$$[\tau]_q = \begin{cases} \dfrac{1-q^\tau}{1-q} & \tau \in \mathbb{C} \\ 1 + \sum_{j=1}^{n-1} q^j & (n \in \mathbb{N}) \end{cases}$$

(1.15)Using the definition (1.15), the q-factorial $[n]_q!$ is given by

$$[n]_q! := \begin{cases} 1 & (n = 0) \\ \prod_{j=1}^{\infty} [j]_q & (n \in \mathbb{N}) \end{cases}$$

For $\tau \in \mathbb{C}$, we shall also make use of the following notation for the basic (or q-) Pochhammer symbol defined above in (1.14):

$$(q^\tau ; q)_n = \begin{cases} 1 & (n = 0) \\ (1-q^\tau)(1-q^{\tau+1}) \dots (1-q^{\tau+n-1}) & (n \in \mathbb{N}) \end{cases}$$

and, for convenience, we write

$$[\tau]_{q,n} := \begin{cases} 1 & (n = 0) \\ \dfrac{(q^\tau ; q)_n}{(1-q)^n} = \prod_{j=1}^{n} [\tau + j - 1]_q & (n \in \mathbb{N}). \end{cases}$$

(1.16)

in terms of the q-numbers $[\tau]_q$ defined by (1.15). Clearly, from the definition (1.16), it is easy to see for the familiar Pocchammer symbol $(\tau)_n$ that

$$\lim_{q \to 1^-} \left\{[\tau]_{q,n}\right\} = \lim_{q \to 1^-} \left\{\frac{(q^\tau ; q)_n}{(1-q)^n}\right\} = (\tau)_n$$

and, for the classical (Euler's) gamma function $\Gamma(z)$, we have

$$\lim_{q \to 1^-} \{\Gamma_q(z)\} = \Gamma(z).$$

For $0 < q < 1$ and the function $\mathcal{Q}(\tau;\, z) f(z)$ given by (1.13), when we apply the q-derivative operator D_q defined by (see [9] and [18])

$$D_q(f(z)) := \begin{cases} \dfrac{f(z) - f(qz)}{1-q} & (0 < q < 1) \\ f'(z) & (q \to 1^-), \end{cases}$$

we get

$$D_q(\mathcal{Q}(\tau;z) f(z)) := \frac{\mathcal{Q}(\tau;z) f(z) - \mathcal{Q}(\tau;z) f(qz)}{(1-q)z}$$

$$= 1 + \sum_{j=2}^{\infty} [j]_q \frac{[\tau(j-1)]^{j-2} e^{-\tau(j-1)}}{(j-1)!} a_j z^{j-1} \; (0 < \tau \leq 1; z \in \mathfrak{U}),$$

where the function $f(z)$ is given by (1.1).

Definition 2. [26] For $\alpha > -1$ and $0 < q < 1$, the linear operator $\mathcal{Q}_\tau^{\alpha,q}$ for functions $f: \mathcal{A} \to \mathcal{A}$ is defined as follows:

$$\mathcal{Q}_\tau^{\alpha,q} f(z) * \mathcal{N}_{q,\alpha+1}(z) = z D_q(\mathcal{Q}(\tau;z) f(z)) \; (z \in \mathfrak{U}),$$

where the function $\mathcal{N}_{q,\alpha+1}$ is given by

$$\mathcal{N}_{q,\alpha+1}(z) := z + \sum_{j=2}^{\infty} \frac{[\alpha + 1]_{q,j-1}}{[j - 1]_q!} \; z^j \;\; (z \in \mathfrak{U}).$$

A simple computation shows that

$$\mathcal{Q}_\tau^{\alpha,q} f(z) := z + \sum_{j=2}^{\infty} \frac{[j]_q! \, [\tau(j-1)]^{j-2} e^{-\tau(j-1)}}{[\alpha + 1]_{q,j-1} (j-1)!} \; a_j \, z^j$$

$$= z + \sum_{j=2}^{\infty} \delta_j \, a_j \, z^j \;\; (0 < \tau < 1; \alpha > -1; 0 < q < 1; z \in \mathfrak{U}),$$

(1.17)

where

$$\delta_j = \frac{[j]_q! \, [\tau(j-1)]^{j-2} e^{-\tau(j-1)}}{[\alpha + 1]_{q,j-1} (j-1)!}. \qquad (1.18)$$

We also note that

$$\delta_2 = \frac{[2]_q! e^{-\tau}}{[\alpha+1]_{q,1}} \text{ and } \delta_3 = \frac{[3]_q! \tau e^{-2\tau}}{[\alpha+1]_{q,2}}. \qquad (1.19)$$

From the equation (1.17), we can easily verify that each of the following relations holds true for all $f \in \mathcal{A}$:

$$[\alpha + 1]_q \mathcal{Q}_\tau^{\alpha,q} f(z) = [\alpha]_q \mathcal{Q}_\tau^{\alpha+1,q} f(z) + q^\alpha z D_q \left(\mathcal{Q}_\tau^{\alpha+1,q} f(z) \right), z \in \mathfrak{U}$$

and

$$\mathcal{R}_\tau^\alpha f(z) := \lim_{q \to 1^-} \{ \mathcal{Q}_\tau^{\alpha,q} f(z) \} = z + \sum_{j=2}^{\infty} \frac{j [\tau(j-1)]^{j-2} e^{-\tau(j-1)}}{(\alpha+1)_{j-1}} \; a_j \, z^j$$

$$= z + \sum_{j=2}^{\infty} \Omega_j a_j z^j \;\; (z \in \mathfrak{U}),$$

where

$$\Omega_j = \frac{j [\tau(j-1)]^{j-2} e^{-\tau(j-1)}}{(\alpha+1)_{j-1}}. \qquad (1.20)$$

Definition 3. A function $f \in \mathcal{A}$ is said to be in the class $\kappa_s^{\alpha,\tau,q}(A,B)$ if and only if

$$\left(\frac{2z \left(\mathcal{Q}_\tau^{\alpha,q} f(z) \right)'}{\mathcal{Q}_\tau^{\alpha,q} g(z) - \mathcal{Q}_\tau^{\alpha,q} g(-z)} \right) \prec \frac{1 + Az}{1 + Bz} \qquad (1.21)$$

$$(-1 \le B < A \le 1; \; 0 < \tau \le 1; \; \alpha > -1; \; 0 < q < 1).$$

Upon letting $q \to 1^-$ in the class $\kappa_s^{\alpha,\tau,q}(A,B)$, we have

$$\lim_{q \to 1^-} \{ \kappa_s^{\alpha,\tau,q}(A,B) \} = \mathcal{G}_s^{\alpha,\tau}(A,B),$$

where

$$\mathcal{G}_s^{\alpha,\tau}(A,B) := \left\{ f : f \in \mathcal{A} \text{ and } \frac{2z \left(\mathcal{R}_\tau^\alpha f(z) \right)'}{\mathcal{R}_\tau^\alpha g(z) - \mathcal{R}_\tau^\alpha g(-z)} \prec \frac{1 + Az}{1 + Bz} \right\}$$

(1.22)

for $(-1 \le B < A \le 1; \; 0 < \tau \le 1; \; \alpha > -1)$.

Definition 4. A function $f \in \mathcal{A}$ is said to be in the class $\kappa_c^{\alpha,\tau,q}(A,B)$ if and only if

$$\left(\frac{2z\left(\mathcal{Q}_{\tau}^{\alpha,q}f(z)\right)'}{\mathcal{Q}_{\tau}^{\alpha,q}g(z)+\mathcal{Q}_{\tau}^{\alpha,q}\overline{g(\bar{z})}}\right)\prec\frac{1+Az}{1+Bz} \qquad (1.23)$$

$(-1 \le B < A \le 1;\ 0 < \tau \le 1;\ \alpha > -1;\ 0 < q < 1)$.If we let $q \to 1^{-}$ in the class $\kappa_{c}^{\alpha,\tau,q}(A,B)$, we have

$$\lim_{q\to 1^{-}}\{\kappa_{c}^{\alpha,\tau,q}(A,B)\}=\mathcal{G}_{c}^{\alpha,\tau}(A,B),$$

where

$$\mathcal{G}_{c}^{\alpha,\tau}(A,B):=\left\{f: f\in\mathcal{A} \text{ and } \frac{2z\left(\mathcal{R}_{\tau}^{\alpha}f(z)\right)'}{\mathcal{R}_{\tau}^{\alpha}g(z)+\mathcal{R}_{\tau}^{\alpha}\overline{g(\bar{z})}}\prec\frac{1+Az}{1+Bz}\right\}$$

(1.24)

for $(-1 \le B < A \le 1;\ 0 < \tau \le 1;\ \alpha > -1)$.

2. SOME PRELIMINARY LEMMAS

The following lemmas will be needed to prove our results.

Lemma 1. (see [8], Lemma 2) If $P(z)$ is given by

$$P(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \in P[A,B],$$

hen

$$|p_n| \le A - B \qquad (n \in \mathbb{N}, -1 \le B < A \le 1).$$

(2.1)

For the coefficient inequalities of the clasees $M_s(\alpha,A,B)$ and $M_c(\alpha,A,B)$ is given by (see [20])

Lemma 2. ([20], Theorem 3.1) Let $g(z) = z + \sum_{j=2}^{\infty} b_j z^j \in M_s(\alpha,A,B)$. Then for $n \ge 1, 0 \le \alpha \le 1$

$$|b_{2n}| \le \frac{(A-B)}{n!\,2^n(1+(2n-1)\alpha)}\prod_{j=1}^{n-1}(A-B+2j),$$

and

$$|b_{2n+1}| \le \frac{(A-B)}{n!\,2^n(1+2n\alpha)}\prod_{j=1}^{n-1}(A-B+2j).$$

Lemma 3. ([20], Theorem 3.2) Let $g(z) = z + \sum_{j=2}^{\infty} b_j z^j \in M_c(\alpha,A,B)$. Then for $n \ge 1, 0 \le \alpha \le 1$

$$|b_{2n}| \le \frac{(A-B)}{(2n-1)!\,(1+(2n-1)\alpha)}\prod_{j=1}^{2n-2}(A-B+j),$$

and

$$|b_{2n+1}| \le \frac{(A-B)}{(2n)!\,(1+2n\alpha)}\prod_{j=1}^{2n-1}(A-B+j).$$

Note: Using the techniques used by A.T. Oladipo [16], we prove the next result below.

3. MAIN RESULT

In this section we give the coefficient inequalities for classes $\kappa_{s}^{\alpha,\tau,q}(A,B)$ and $\kappa_{c}^{\alpha,\tau,q}(A,B)$.

Theorem 1. Let $f(z) \in \kappa_{s}^{\alpha,\tau,q}(A,B)$. Then, for all $n \ge 1$,

$$|a_2| \leq \frac{(A-B)}{2\delta_2},$$
$$|a_3| \leq \frac{(A-B)(2+\delta_3)}{3!\,\delta_3},$$
$$|a_4| \leq \frac{(A-B)[3(2+\delta_3(A-B))]}{4!\,\delta_4} \qquad (3.1)$$

and

$$|a_5| \leq \frac{(A-B)[4(2+\delta_3(A-B))+\delta_5(A-B+2)]}{40\delta_5}$$

Proof. Since $g(z) \in S_s^*(A,B)$, then we have from Lemma 2 when $\alpha = 0$

$$|b_2| \leq \frac{A-B}{2},\ |b_3| \leq \frac{A-B}{2} \qquad (3.2)$$
$$|b_4| \leq \frac{(A-B)(A-B+2)}{8},\ |b_5| \leq \frac{(A-B)(A-B+2)}{8}$$

Since $f(z) \in \kappa_s^{\alpha,\tau,q}(A,B)$, it follows from (1.21) by using Equation (1.6)

$$\left(2z\left(Q_\tau^{\alpha,q} f(z)\right)'\right) = \left(Q_\tau^{\alpha,q} g(z) - Q_\tau^{\alpha,q} g(-z)\right)\left(1+\sum_{j=1}^{\infty} p_j z^j\right)$$
$$(0 < \tau \leq 1; \alpha > -1; 0 < q < 1). \qquad (3.3)$$

It follows that

$$z + 2\delta_2 a_2 z^2 + 3\delta_3 a_3 z^3 + 4\delta_4 a_4 z^4 + 5\delta_5 a_5 z^5 + \cdots + 2n\delta_{2n} a_{2n} z^{2n} + (2n+1)\delta_{2n+1} a_{2n+1} z^{2n+1} + \cdots$$
$$= \left(z + \delta_3 b_3 z^3 + \delta_5 b_5 z^5 + \cdots + \delta_{2n-1} b_{2n-1} z^{2n-1} + \delta_{2n+1} b_{2n+1} z^{2n+1} + \cdots\right)\left(1 + p_1 z + p_2 z^2 + p_3 z^3 + p_4 z^4 + \cdots\right).$$

Equating the coefficients of like powers of z, we have

$$2\delta_2 a_2 = p_1,$$
$$\delta_3(3a_3 - b_3) = p_2,$$
$$4\delta_4 a_4 = \delta_3 b_3 p_1 + p_3,$$
$$\delta_5(5a_5 - b_5) = \delta_3 b_3 p_2 + p_4. \qquad (3.4)$$

By applying (3.2) and followed by Lemma 1, we get (3.1) from (3.4). This completes the proof. Letting $q \to 1^-$ in Theorem 1, we obtain the following corollary.

Corollary 1. Let $f(z) \in \mathcal{G}_s^{\alpha,\tau}(A,B)$. Then, for all $n \geq 1$,

$$|a_2| \leq \frac{(A-B)}{2!\,\Omega_2},$$
$$|a_3| \leq \frac{(A-B)(2+\Omega_3)}{3!\,\Omega_3},$$
$$|a_4| \leq \frac{(A-B)[3(2+\Omega_3(A-B))]}{4!\,\Omega_4}, \qquad (3.5)$$

and

$$|a_5| \leq \frac{(A-B)[4(2+\Omega_3(A-B))+\Omega_5(A-B+2)]}{40\Omega_5}.$$

where $\Omega_j \forall (j = 2,3,4,5)$ are given by (1.20).

Theorem 2. Let $f(z) \in \kappa_c^{\alpha,\tau,q}(A,B)$. Then, for all

$$|a_2| \le \frac{(A-B)(\delta_2+1)}{2!\,\delta_2}$$

$$|a_3| \le \frac{(A-B)[2+2\delta_2(A-B)+3\delta_3(A-B+1)]}{3!\,\delta_3},$$

$$|a_4| \le \frac{(A-B)[6+6\delta_2(A-B)+3\delta_3(A-B)(A-B+1)+\delta_4(A-B+1)(A-B+2)]}{4!\,\delta_4},$$

$n \ge 1$,

and (3.6)

$$|a_5| \le \frac{(A-B)}{5!\,\delta_5}[24+24\delta_2(A-B)+12\delta_3(A-B)(A-B+1)+4\delta_4(A-B)(A-B+1)(A-B+2) + \delta_5(A-B+1)(A-B+2)(A-B+3)]$$

Proof. Since $g(z) \in S_c^*(A,B)$, then we have from Lemma 3 when $\alpha = 0$

$$|b_2| \le (A-B),\ |b_3| \le \frac{(A-B)(A-B+1)}{2},$$

$$|b_4| \le \frac{(A-B)[(A-B+1)(A-B+2)]}{6},$$

$$|b_5| \le \frac{(A-B)[(A-B+1)(A-B+2)(A-B+3)]}{24}. \quad (3.7)$$

Since $f(z) \in \kappa_c^{\alpha,\tau,q}(A,B)$, it follows from (1.23) by using Equation (1.6)

$$\left(2z\left(Q_\tau^{\alpha,q} f(z)\right)'\right) = \left(Q_\tau^{\alpha,q} g(z) + Q_\tau^{\alpha,q}\overline{g(\bar{z})}\right)\left(1+\sum_{j=1}^{\infty} p_j z^j\right)$$

$$(0 < \tau \le 1; \alpha > -1; 0 < q < 1).$$

(3.8)

It follows that

$$z + 2\delta_2 a_2 z^2 + 3\delta_3 a_3 z^3 + 4\delta_4 a_4 z^4 + 5\delta_5 a_5 z^5 + \cdots + 2n\delta_{2n} a_{2n} z^{2n} + (2n+1)\delta_{2n+1} a_{2n+1} z^{2n+1} + \cdots$$
$$= \left(z + \delta_2 b_2 z^2 + \delta_3 b_3 z^3 + \delta_4 b_4 z^4 + \delta_5 b_5 z^5 + \cdots + \delta_{2n} b_{2n} z^{2n} + \delta_{2n+1} b_{2n+1} z^{2n+1}\right)$$
$$.\left(1 + p_1 z + p_2 z^2 + p_3 z^3 + p_4 z^4 + \cdots\right).$$

Equating the coefficients of like powers of z, we have

$$\delta_2(2a_2 - b_2) = p_1$$
$$\delta_3(3a_3 - b_3) = \delta_2 b_2 p_1 + p_2$$
$$\delta_4(4a_4 - b_4) = \delta_3 b_3 p_1 + \delta_2 b_2 p_2 + p_3,$$
$$\delta_5(5a_5 - b_5) = \delta_4 b_4 p_1 + \delta_3 b_3 p_2 + \delta_2 b_2 p_3 + p_4. \quad (3.9)$$

By applying (3.7) and followed by Lemma 1, we get (3.6) from (3.9). This completes the proof.

Letting $q \to 1^-$ in Theorem 2, we obtain the following corollary.

Corollary 2. Let $f(z) \in \mathcal{G}_c^{\alpha,\tau}(A,B)$. Then, for all $n > 1$,

$$|a_2| \leq \frac{(A-B)(\Omega_2+1)}{2!\,\Omega_2}$$

$$|a_3| \leq \frac{(A-B)[2+2\Omega_2(A-B)+3\Omega_3(A-B+1)]}{3!\,\Omega_3},$$

$$|a_4| \leq \frac{(A-B)[6+6\Omega_2(A-B)+3\Omega_3(A-B)(A-B+1)+\Omega_4(A-B+1)(A-B+2)]}{4!\,\Omega_4}, \text{and}$$

$$|a_5| \leq \frac{(A-B)}{5!\,\Omega_5}[(24+24\Omega_2(A-B)+12\Omega_3(A-B)(A-B+1)+4\Omega_4(A-B)(A-B+1)(A-B+2)) +\Omega_5(A-B+1)(A-B+2)(A-B+3)]. \quad (3.10)$$

where $\Omega_j \forall (j = 2,3,4,5)$ are given by (1.20). Our next result is to briefly look at the connection of our classes to the classical Fekete-Szegö Theorem.

Theorem 3. Let $f(z) \in \kappa_s^{\alpha,\tau,q}(A,B)$. Then,

$$|a_3 - \mu a_2^2| \leq \frac{(A-B)\left[4\delta_2^2 + 2\delta_3\delta_2^2 - 3\mu\delta_3(A-B)\right]}{12\delta_3\delta_2^2} \quad \mu \leq 0, \quad (3.11)$$

and

$$|a_2a_4 - a_3^2| \leq (A-B)^2 \left\{\frac{9\delta_3^2(\delta_3(A-B)+2) - 4\delta_2\delta_4(2+\delta_3)^2}{144\delta_2\delta_3^2\delta_4}\right\}. \quad (3.12)$$

Proof. Also, the proof could be obtained from Theorem 1.

Letting $q \to 1^-$in Theorem 3, we get the next corollary.

Corollary 3. Let $f(z) \in \mathcal{G}_s^{\alpha,\tau}(A,B)$. Then,

$$|a_3 - \mu a_2^2| \leq \frac{(A-B)[4\Omega_2^2 + 2\Omega_3\Omega_2^2 - 3\mu\Omega_3(A-B)]}{12\Omega_3\Omega_2^2},$$

(3.13)

and

$$|a_2a_4 - a_3^2| \leq (A-B)^2 \left\{\frac{9\Omega_3^2(\Omega_3(A-B)+2) - 4\Omega_2\Omega_4(2+\Omega_3)^2}{144\Omega_2\Omega_3^2\Omega_4}\right\}.$$

(3.14)

Theorem 4. Let $f(z) \in \kappa_c^{\alpha,\tau,q}(A,B)$. Then, for $\mu \leq 0$,

$$|a_3 - \mu a_2^2| \leq \frac{(A-B)\left[4\delta_2^2[2+2\delta_2(A-B)+3\delta_3(A-B+1)] - 6\mu\delta_3(A-B)(\delta_2+1)^2\right]}{24\delta_3\delta_2^2},$$

(3.15)

and

$$|a_2a_4 - a_3^2| \leq \frac{(A-B)^2}{144\delta_2\delta_3^2\delta_4}\{3\delta_3^2(\delta_2+1)[6+6\delta_2(A-B)+3\delta_3(A-B)(A-B+1)+ \delta_4(A-B+1)(A-B+2)] - 4\delta_2\delta_4[2+2\delta_2(A-B)+3\delta_3(A-B+1)]^2\}.$$

(3.16)

Proof. Also, the proof could be obtained from Theorem 2.

Letting $q \to 1^-$ in Theorem 4, we get the next corollary.

Corollary 4. Let $f(z) \in \mathcal{G}_c^{\alpha,\tau}(A,B)$. Then,

$$|a_3-\mu a_2^2|\le\frac{(A-B)[4\Omega_2^2[2+2\Omega_2(A-B)+3\Omega_3(A-B+1)]-6\mu\Omega_3(A-B)(\Omega_2+1)^2]}{24\Omega_3\Omega_2^2},$$

(3.17)

and

$$|a_2a_4-a_3^2|\le\frac{(A-B)^2}{144\delta_2\delta_3^2\delta_4}\{3\Omega_3^2(\Omega_2+1)[6+6\Omega_2(A-B)+3\Omega_3(A-B)(A-B+1)+\Omega_4(A-B+1)(A-B+2)]-4\Omega_2\Omega_4[2+2\Omega_2(A-B)+3\Omega_3(A-B+1)]^2\}.$$

(3.18)

In the finally result is on sufficient condition for a function $f(z)$ to be in $\kappa_s^{\alpha,\tau,q}\ (A,B)$.

Theorem 5. Let the function $f(z)$ defined by (1.1) and let

$$\sum_{j=2}^{\infty}\delta_j\{|(2ja_j-(1-(-1)^j)b_j)|-|(1-(-1)^j)Ab_j-2jBa_j\ |\}\le 2(A-B)$$

(3.19)

holds, then $f(z)$ belong to $\kappa_s^{\alpha,\tau,q}(A,B)$.

Proof. Assume that the inequality (1.21) holds. Then we get for $z\in\mathfrak{U}$.

$$\left|\sum_{j=2}^{\infty}2j\delta_ja_jz^j-\sum_{j=2}^{\infty}(1-(-1)^j)b_j\delta_jz^j\right|=\left|\sum_{j=2}^{\infty}(1-(-1)^j)Ab_j\delta_jz^j-\sum_{j=2}^{\infty}2jBa_j\delta_jz^j\right|+2|(A-B)z|$$

$$\left|\sum_{j=2}^{\infty}(2ja_j-(1-(-1)^j)b_j)\delta_jz^j\right|=\left|\sum_{j=2}^{\infty}((1-(-1)^j)Ab_j-2jBa_j)\delta_jz^j\right|+|2(A-B)z|$$

$$\le$$

$$\sum_{j=2}^{\infty}\delta_j|(2ja_j-(1-(-1)^j)b_j)|r^j=\sum_{j=2}^{\infty}\delta_j\left|((1-(-1)^j)Ab_j-2jBa_j)\right|r^j+2(A-B)r,\ (0<r<1).$$

Now, letting $r\to 1$, therefore, we obtain

$$\sum_{j=2}^{\infty}\delta_j\{|(2ja_j-(1-(-1)^j)b_j)|-|(1-(-1)^j)Ab_j-2jBa_j|\}\le 2(A-B).$$

Therefore, it follows that

$$\left|\frac{2z\left(Q_\tau^{\alpha,q}f(z)\right)'-\left(Q_\tau^{\alpha,q}g(z)-Q_\tau^{\alpha,q}g(-z)\right)}{A\left(Q_\tau^{\alpha,q}g(z)-Q_\tau^{\alpha,q}g(-z)\right)-B\left(2z\left(Q_\tau^{\alpha,q}f(z)\right)'\right)}\right|<1,z\in\mathfrak{U}.$$

(3.20)

Assuming

$$|w(z)|=\left|\frac{2z\left(Q_\tau^{\alpha,q}f(z)\right)'-\left(Q_\tau^{\alpha,q}g(z)-Q_\tau^{\alpha,q}g(-z)\right)}{A\left(Q_\tau^{\alpha,q}g(z)-Q_\tau^{\alpha,q}g(-z)\right)-B\left(2z\left(Q_\tau^{\alpha,q}f(z)\right)'\right)}\right|<1,z\in\mathfrak{U}.$$

(3.21)

Then $w(0)=0, w(z)$ is analytic in $|z|<1$ and $|w(z)|<1$. Therefore, $f(z)\in\kappa_s^{\alpha,\tau,q}(A,B)$.

Theorem 6. Let the function $f(z)$ defined by (1.1) and let

$$\sum_{j=2}^{\infty} \delta_j\{|(ja_j - b_j)| - |Ab_j - Ba_j\ |\} \leq (A - B) \tag{3.22}$$

holds, then $f(z)$ belong to $\kappa_c^{\alpha,\tau,q}(A, B)$.

Proof. Assume that the inequality (1.23) holds. Then we get for $z \in \mathfrak{U}$.

$$\left|\sum_{j=2}^{\infty} j\delta_j a_j z^j - \sum_{j=2}^{\infty} b_j \delta_j z^j\right| = \left|\sum_{j=2}^{\infty} Ab_j\delta_j z^j - \sum_{j=2}^{\infty} jBa_j\delta_j z^j\right| + |(A - B)z|$$

$$\left|\sum_{j=2}^{\infty} (ja_j - b_j)\delta_j z^j\right| = \left|\sum_{j=2}^{\infty} (Ab_j - jBa_j)\delta_j z^j\right| + |(A - B)z|$$

$$\leq$$

$$\sum_{j=2}^{\infty} \delta_j|(ja_j - b_j)|r^j = \sum_{j=2}^{\infty} \delta_j|(Ab_j - jBa_j)|r^j + (A - B)r,\ (0 < r < 1).$$

Now, letting $r \to 1$, therefore, we obtain

$$\sum_{j=2}^{\infty} \delta_j\{|(ja_j - b_j)| - |Ab_j - jBa_j\ |\} \leq (A - B).$$

Therefore, it follows that

$$\left|\frac{2z\left(Q_\tau^{\alpha,q} f(z)\right)' - \left(Q_\tau^{\alpha,q} g(z) + Q_\tau^{\alpha,q}\overline{g(\bar{z})}\right)}{A\left(Q_\tau^{\alpha,q} g(z) + Q_\tau^{\alpha,q}\overline{g(\bar{z})}\right) - B\left(2z\left(Q_\tau^{\alpha,q} f(z)\right)'\right)}\right| < 1, z \in \mathfrak{U}.$$

(3.23)

Assuming

$$|w(z)| = \left|\frac{2z\left(Q_\tau^{\alpha,q} f(z)\right)' - \left(Q_\tau^{\alpha,q} g(z) + Q_\tau^{\alpha,q}\overline{g(\bar{z})}\right)}{A\left(Q_\tau^{\alpha,q} g(z) + Q_\tau^{\alpha,q}\overline{g(\bar{z})}\right) - B\left(2z\left(Q_\tau^{\alpha,q} f(z)\right)'\right)}\right| < 1, z \in \mathfrak{U}.$$

(3.24)

Then $w(0) = 0, w(z)$ is analytic in $|z| < 1$ and $|w(z)| < 1$. Therefore, $f(z) \in \kappa_c^{\alpha,\tau,q}(A, B)$.

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