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Galerkin-Based Approximate Solution for Heat Transfer Problems

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ABSTRACT: This paper presents the application of the Weighted Residuals Method, specifically the Least Squares Method, to solve the steady-state one-dimensional heat conduction problem in a slab with thermal conductivity linearly dependent on temperature. The proposed solution, a fourth-degree polynomial derived over the domain, demonstrates notable accuracy despite its simplicity, as evidenced by the RMS error of 0.0014328220965349. Additionally, if higher accuracy is desired, the Method of Weighted Residuals allows for the incorporation of more subdomains and the use of higher-degree polynomials.

KEYWORDS: Nonlinear differential equations, boundary value problems, heat problems. Numerical analysis

I. INTRODUCTION

Thermodynamics is a discipline that falls within physics and is dedicated to the study of phenomena related to heat. The interest of thermodynamics focuses especially on considering the way in which different forms of energy are transformed and the relationship between these processes and temperature. In fact, there are evaluations that establish that the development of the discipline was done alongside an attempt to achieve greater efficiency in the use of machines, efficiency that implied that the least amount of energy was lost in the form of heat.

Thermodynamics is governed by a set of laws that describe the behaviour of energy. The first of these is the principle of energy conservation, which states that energy cannot be created or destroyed, but only transformed from one form to another. In this sense, heat is simply a form of energy that can be derived from others, such as work. The second law of thermodynamics dictates that in a closed system, entropy increases, where entropy is understood as a process of disorder in which energy becomes unavailable for work. Finally, the third law of thermodynamics asserts that it is impossible to reach absolute zero in a system through a finite number of steps.

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The primary aim of this article is to develop a practical analytical approximation for the boundary value problem related to the steady-state, one-dimensional heat conduction within a slab. Importantly, the thermal conductivity of the slab is assumed to vary linearly with temperature. Given the fundamental importance of heat transfer phenomena, both in theoretical studies and practical applications such as equipment design and operation, there is a pressing need to explore analytical approximate solutions to the governing equations. This effort aligns with the broader research objectives and highlights the necessity of developing efficient approaches to tackle heat transfer problems. There are various semi-analytical methods available for addressing nonlinear differential equations, including the Power Series Extender Method (PSEM) [3], the Homotopy Perturbation Method (HPM) [4], Homotopy Analysis Method (HAM) [5], Adomian's Decomposition Method [6], the Modified Taylor Series Method (MTM) [7-10], the Perturbation Method (PM) [11], and the Method of Weighted Residuals (MWR) [12,13,14], among others. In this study, it is assumed that the exact solution for the one-dimensional (1-D) steady-state heat conduction problem is obtained using Maple 2021. Furthermore, the root-mean-squared (RMS) error is employed as a key metric to assess the accuracy of the proposed approximations.

The paper is organized as follows: Section II outlines the basic principles of the Weighted Residuals Method. Section III describes the case study, Section IV discusses the results, and Section V provides the conclusions

II. WEIGHTED RESIDUALS METHOD

To introduce MWR, let us consider a general boundary value problem whose governing differential equation is presented as follows:

$$Lu(x) = 0, x \in \Omega,$$
$$u(x) = g(x), x \in \Gamma, \tag{1}$$

where L represents a differential operator, $u = u(x)$ denotes the dependent variable defined within a region Ω with boundary Γ, and x refers to the spatial coordinates. In the Method of Weighted Residuals (MWR), the goal is to approximate the solution $u(x)$ of Equation (1) by using a trial solution $u_n(x)$, which is chosen in a specific way. However, this trial solution generally does not satisfy the governing differential equation. As a result, substituting the trial solution into the governing equation leads to a residual, represented by R [12-14]. To achieve the "best" solution, efforts are made to distribute this residual over the region Ω by minimizing the integral of the residual across Ω. This can be written as:"

$$Minimize = \int_{\Omega} R d\Omega. \tag{2}$$

The range of possibilities to accomplish this goal can be broadened by ensuring that a weighted residual is minimized over the entire region of interest. By applying a weighting function, it becomes possible to attain a minimum value of zero for the weighted integral. Letting the weighting functions be denoted by w, the desired objective of the MWR is then defined as follows

$$\int_{\Omega} wR d\Omega = 0. \tag{3}$$

The idea of approximating the solution $u(x)$ of a differential equation using trial solutions is well-established. However, the successful application of MWR heavily depends on the proper selection of the trial solution. This choice is powerful because it allows for the inclusion of known information about the problem into the trial solution. In lower-order approximations (i.e., for small n in $u_n(x)$), this selection can have a significant impact on the accuracy of the results. In higher-order approximations, it can affect the convergence of the method [12-14]. Among the various trial solutions used by different researchers, perhaps polynomial series such as

$$u_n(x) = \sum_{i=1}^{n} c_i N(x) = \sum_{i=1}^{n} c_i x^i. \tag{4}$$

Polynomials are the most commonly used choice for this purpose. In equation (4), c_i are arbitrary constants that must be determined during the minimization process outlined in equation (3). The functions $N(x)$ are preselected and referred to as trial functions or shape functions. The widespread preference for polynomials arises mainly from their simplicity in manipulation. Additionally, the weighting functions can be chosen in various ways, with each selection corresponding to a different MWR criterion [12-14].

Subdomains method. In this process, the Ω domain is divided into m smaller subdomains $\Omega_j, j = 1,2,\cdots,m$, which are not necessarily disjoint. The weights are selected as follows:

$$w_j = \begin{cases} 1, x \in \Omega_j, \\ 0, x \notin \Omega_j, \end{cases} \tag{5}$$

and

$$\int_{\Omega_j} R d\Omega_j = 0, j = 1,2,\cdots m. \tag{6}$$

As m increases, the integral of the differential equation over each subdomain approaches zero. As a result, the equation is increasingly satisfied, on average, in progressively smaller domains, eventually tending to zero across the entire domain [12-14].

Colocation Method. In this method, the weighting functions w_j are chosen to be the displaced Dirac Delta functions

$$w_j = \delta_j = \delta(x - x_j). \tag{7}$$

Now (2) is given by

$$\int_{\Omega_j} w_j R d\Omega = \int \delta_j R d\Omega = R_j = 0, j = 1,2,\cdots m, \tag{8}$$

where R_j represents the value of R evaluated at the point x_j. As a result, the residual is forced to vanish at m specified collocation points, $x_j = 1,2,\cdots m$.. As m increases, the residual vanishes at an increasing number of points, presumably approaching zero everywhere.

Least Squares Method. In this method, the weighting functions w_j are choosen to be

$$w_j = \frac{\partial R}{\partial c_j}. \tag{9}$$

Now Eq. (2) is given by

$$\frac{\partial}{\partial c_j} \int_{\Omega} R^2 d\Omega = 2 \int_{\Omega} \frac{\partial R}{\partial c_j} R d\Omega = 0, j = 1,2,\cdots n. \tag{10}$$

The integral of the square of the residual is minimized with respect to the undetermined parameters to provide N simultaneous equations for the c_j's.

Method of moments. In this method, the weighting functions w_j are choosen to be

$$w_j = P_j(x). \tag{11}$$

where $P_j(x)$are orthogonal polynomials defined over the domain Ω. This approach is especially beneficial in one-dimensional problems, where the theory of orthogonal polynomials is well-established. In these problems, the common use of weighting functions, denoted by $w(x)$, results in the following:

$$\int_{\Omega} x^j R d\Omega = 0. \tag{12}$$

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The form of Equation (12) led to the term 'method of moments.' However, it is important to note that the set $\{w_j\} = \{x^j\} = \{1, x, x^2, \cdots\}$ is not orthogonal over the interval $0 \le x \le 1$, and generally, better results can be obtained by orthogonalizing them prior to application [12-14].

Galerkin method. In this method, the weighting functions $w_j$are choosen to be identical to the shape functions N_j themselves, that is,

$$w_j = N_j(x), j = 1,2, \cdots, m. \tag{14}$$

Therefore, Eq. (3) is given by

$$\int_\Omega N_j(x) R d\Omega = 0, j = 1,2, \cdots, m.$$

In vector-matrix notation, we have

$$\int_\Omega \boldsymbol{N} R d\Omega = 0, \tag{15}$$

where $\boldsymbol{N} = (N_1, N_2, \cdots, N_m)^T$.. By leveraging the well-established principle that a continuous function is zero if it is orthogonal to every member of a set, it becomes clear that the Galerkin method forces the residual to vanish by ensuring its orthogonality to each member of a complete set of basis functions [12-14].

III. CASE STUDY

This article presents a case study on one-dimensional (1-D) steady-state heat conduction in a slab with linearly temperature-dependent thermal conductivity [9]; see Fig. 1.

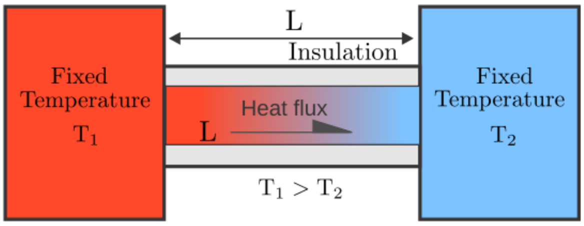

Figure 1. The 1-D conduction of heat through an insulated slab.

The non-dimensionalization process for this problem was outlined in [9], which also proposed an approximate solution using the Modified Taylor Series Method (MTSM). In the present study, we aim to obtain an approximate solution by employing the Method of Weighted Residuals (MWR), specifically the method of moments. The differential equation for this case study is expressed as follows:

$$\frac{d^2y}{dz^2} + \varepsilon y \frac{d^2}{dz^2} + \varepsilon \left(\frac{dy}{dz}\right)^2 = 0, \tag{16}$$

with boundary conditions given by

$$y(0) = 1, y(1) = 0. \tag{17}$$

For the case where $\varepsilon = 1$, the exact solution derived using Maple 2021 is given as:

$$y_E = -1 + \sqrt{-3x + 4}. \tag{18}$$

Applying the ***least square method***, we have the domain $0 \leq x \leq 1$. Additionally, the proposed solution is expressed as a fourth-degree polynomial, given by:

$$y = ex^4 + dx^3 + cx^2 + bx + a. \tag{19}$$

Substituting the boundary conditions in (19) we have

$$y = ex^4 + dx^3 + cx^2 - (1 + c + d + e)x + 1. \tag{20}$$

Equation (20) is substituted into Equation (16) to calculate the residual R. Subsequently, applying Equation (15) to compute the partial derivatives with respect to c, d, e, and performing the integration over the domain $0 \leq x \leq 1$, results in three equations. The parameters c, d, emust then be determined using a numerical algorithm, such as the Newton-Raphson method, Homotopy continuation method [15-17].

$$\begin{aligned} w_1(x) &= \frac{\partial}{\partial c} R(x), \\ w_2(x) &= \frac{\partial}{\partial d} R(x), \\ w_3(x) &= \frac{\partial}{\partial e} R(x), \\ \int_0^1 & w_1(x) R(x) dx = 0, \\ \int_0^1 & w_2(x) R(x) dx = 0, \\ \int_0^1 & w_3(x) R(x) dx = 0. \end{aligned} \tag{21}$$

Solving equation system (21), the solutions are $c = -0.223097077369708, d = 0.197318323576174, e = -0.223481576090948$. By substituting the values of c, d, e in eq. (20) we have

$$y_G(x) = -0.2234815760915x^4 + 0.1973183235762x^3 - 0.2230970773697x^2 - 0.750739670116x + 1 \tag{22}$$

The solution obtained with MTSM [19] is given by

$$y_T(x) = 1 - 0.768425x - 0.147619245x^2 - 0.056717159x^3 - 0.027239301x^4. \tag{23}$$

Figure 2 provides a comparison between the exact solution (18), the Galerkin method (22), and the Modified Taylor Series Method (MTSM) (23). It is observed that the solution obtained using the Galerkin method (22) demonstrates superior performance.

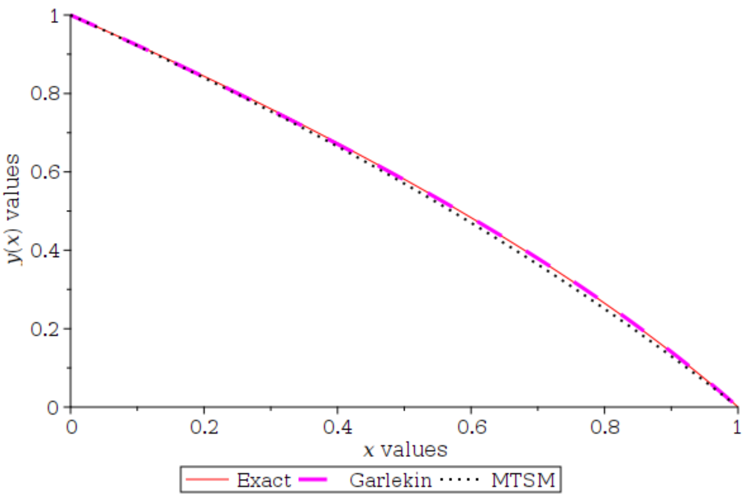

Figure 2. Comparison of exact solutions, Galerkin method vs MTSM.

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IV. DISCUSSION

To solve Equation (16), domain was used. The resulting solution, given by Equation (22), is a fourth-degree polynomial. Figure 2 illustrates a comparison of the absolute errors for Equations (22) and (23). The absolute error of the solution (23), derived using the Method of Galerkin, is significantly smaller. It is worth noting that Equation (23) is also a fourth-degree polynomial, determined using the Modified Taylor Series Method (MTSM) [9]

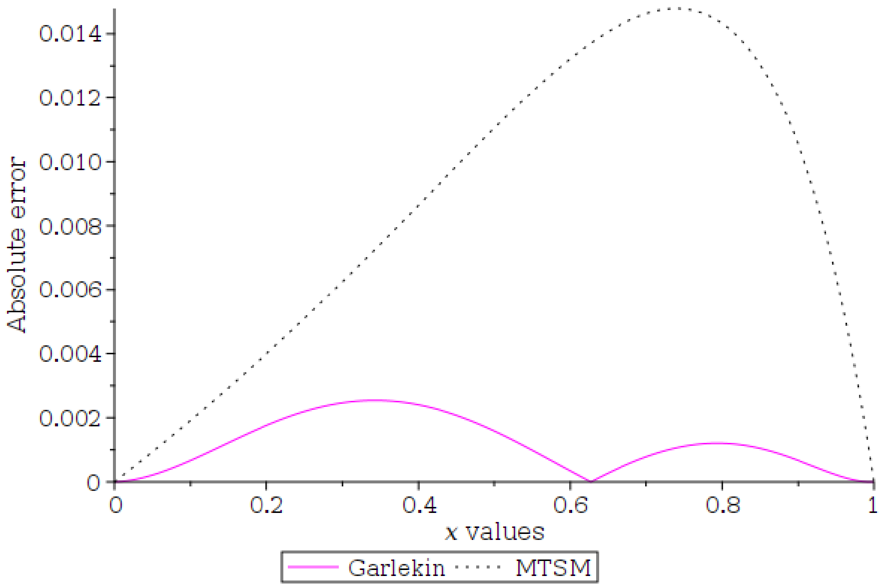

Figure 3. Absolute error for equations (21) and (22).

To measure the RMS error in the interval defined by the boundary conditions, we will use the formula

$$E_{rms} = \sqrt{\frac{1}{b-a}\int_a^b \big(E(t)\big)^2 dt}. \tag{24}$$

Table 1 displays the RMS error for different values of ε, along with the approximate solutions obtained for each value of ε. As ε increases, the RMS error in the approximations also increases. Specifically, when $\varepsilon = 1$, the RMS error for the solution obtained using the Modified Taylor Series Method (MTSM) is 0.00980864863178778. In comparison, the RMS error for the solution in Equation (22) is lower. This indicates that the RMS error using the Galerkin method is 0.00143282209653488, which is 6.846 times smaller than that obtained using the MTSM [9]. Therefore, the solution derived using Galerkin exhibits greater accuracy than the solution obtained using MTSM.

Table 1: RMS error for different ε.

Value ε	Error RMS	Polynomial equation obtained
0.5	0.0001329101394657	$y(x) = -0.0402113361866x^4 - 0.002630459425739x^3 - 0.123834876141x^2 - 0.833323328247x + 1$
1	0.0014328220965349	$y(x) = -0.22348157609x^4 + 0.1973183236x^3 - 0.22309707737x^2 - 0.750739670116x + 1$
1.5	0.003692446990718	$y(x) = -0.04015385866137x^4 - 0.2578655131126x^3 + 0.02187402939203x^2 - 0.7238546576181x + 1$

V. CONCLUSIONS

This study employs the Galerkin method to obtain a polynomial solution for the steady-state one-dimensional heat conduction problem in a slab, where thermal conductivity varies linearly with temperature, subject to Dirichlet boundary conditions. The methodology involves utilizing a trial function integrated over the domain according to a defined procedure. The resulting system of equations is formulated in terms of constants, which are determined using numerical techniques such as the Newton-Raphson method or homotopy continuation methods. The derived polynomial solution is of fourth degree, demonstrating enhanced accuracy. A comparison with other approaches, including the Modified Taylor Series Method (MTSM) reported in the literature, underscores the effectiveness of the Method of Weighted Residuals (MWR) as a reliable and efficient strategy for solving boundary value problems, reducing the reliance on more complex and computationally intensive methods.

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